

The colourful simplicial depth conjecture

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Abstract

Given $d + 1$ sets of points, or colours, $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d , a *colourful simplex* is a set $T \subseteq \bigcup_{i=1}^{d+1} \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$, for all $i \in \{1, \dots, d + 1\}$. The colourful Carathéodory theorem states that, if $\mathbf{0}$ is in the convex hull of each \mathbf{S}_i , then there exists a colourful simplex T containing $\mathbf{0}$ in its convex hull. Deza, Huang, Stephen, and Terlaky (*Colourful simplicial depth*, Discrete Comput. Geom., **35**, 597–604 (2006)) conjectured that, when $|\mathbf{S}_i| = d + 1$ for all $i \in \{1, \dots, d + 1\}$, there are always at least $d^2 + 1$ colourful simplices containing $\mathbf{0}$ in their convex hulls. We prove this conjecture via a combinatorial approach.

Keywords: colourful Carathéodory theorem, colourful simplicial depth, octahedral systems

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1. Introduction

A *colourful point configuration* is a collection of $d+1$ sets of points $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d . A *colourful simplex* is a subset T of $\bigcup_{i=1}^{d+1} \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$. The colourful Carathéodory theorem, proved by Bárány in 1982 [1], states that, given a colourful point configuration $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d such that $\mathbf{0} \in \bigcap_{i=1}^{d+1} \text{conv}(\mathbf{S}_i)$, there exists a colourful simplex T containing $\mathbf{0}$ in its convex hull. In the same paper, Bárány uses this theorem combined with Tverberg's theorem to give a bound on simplicial depth. His argument motivated the following question: how many colourful simplices, at least, contain $\mathbf{0}$ in their convex hulls?

Let $\mu(d)$ denote the minimal number of colourful simplices containing $\mathbf{0}$ in their convex hulls over all colourful point configurations $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ in \mathbb{R}^d such that $\mathbf{0} \in \text{conv}(\mathbf{S}_i)$ and $|\mathbf{S}_i| = d+1$ for $i = 1, \dots, d+1$. The colourful Carathéodory theorem states that $\mu(d) \geq 1$. The quantity $\mu(d)$ has been investigated by Deza et. al [2]. They proved that $2d \leq \mu(d) \leq d^2 + 1$ and conjectured that $\mu(d) = d^2 + 1$. Later I. Bárány and J. Matoušek [3] proved that $\mu(d) \geq \max\left(3d, \left\lceil \frac{d(d+1)}{5} \right\rceil\right)$ for $d \geq 3$, Stephen and Thomas [4] proved that $\mu(d) \geq \left\lfloor \frac{(d+2)^2}{4} \right\rfloor$, and Deza et. al [5] showed that $\mu(d) \geq \left\lceil \frac{(d+1)^2}{2} \right\rceil$. Deza et. al [6] improved the bound to $\frac{1}{2}d^2 + \frac{7}{2}d - 8$ for $d \geq 4$. This latter result was obtained using a combinatorial generalization of the colourful point configurations suggested by Bárány and known as *octahedral systems*, see [5].

We use this combinatorial approach to prove the conjecture.

Theorem 1. *The equality $\mu(d) = d^2 + 1$ holds for every integer $d \geq 1$.*

The outline of the paper goes as follows. Section 2 is divided into two parts. First we define the octahedral systems and show their link with the colourful point configurations. Second, we introduce one of our main tools: the decomposition of an octahedral system over some elementary octahedral systems called umbrellas. Section 3 is devoted to the proof of Theorem 1.

2. Preliminaries

2.1. Octahedral systems

Let V_1, \dots, V_n be n pairwise disjoint finite sets, each of size at least 2. An *octahedral system* is a set $\Omega \subseteq V_1 \times \dots \times V_n$ satisfying the *parity condition*: the cardinality of $\Omega \cap (X_1 \times \dots \times X_n)$ is even if $X_i \subseteq V_i$ and $|X_i| = 2$ for all $i \in \{1, \dots, n\}$. We use the terminology of hypergraphs to describe an octahedral system: the sets V_i are the *classes*, the elements in V_i are the *vertices*, and the n -tuples in $V_1 \times \dots \times V_n$ are the *edges*. An edge whose i th component is a vertex $x \in V_i$ is *incident with the vertex x* , and conversely. A vertex x incident with no edges is *isolated*. A class V_i is *covered* if each vertex of V_i is incident with at least one edge. Finally, the set of edges incident with x is denoted by $\delta_\Omega(x)$ and the *degree of x* , denoted by $\deg_\Omega(x)$, refers to $|\delta_\Omega(x)|$.

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Lemma 1. *In every nonempty octahedral system, at least one class is covered.*

PROOF. Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$. Suppose that no classes are covered. There is at least one isolated vertex x_i in each V_i . Hence, if there were an edge (y_1, \dots, y_n) in Ω , then the parity condition would not be satisfied for $X_i = \{x_i, y_i\}$.

Given a colourful point configuration $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$, the Octahedron Lemma [3, 2] states that, for any $\mathbf{S}'_1 \subseteq \mathbf{S}_1, \dots, \mathbf{S}'_{d+1} \subseteq \mathbf{S}_{d+1}$, with $|\mathbf{S}'_1| = \cdots = |\mathbf{S}'_{d+1}| = 2$, the number of colourful simplices generated by $\bigcup_{i=1}^{d+1} \mathbf{S}'_i$ and containing $\mathbf{0}$ in their convex hulls is even. The hypergraph over $V_1 \times \cdots \times V_n$ where V_i is identified with \mathbf{S}_i and whose edges are identified with the colourful simplices containing $\mathbf{0}$ in their convex hulls is therefore an octahedral system. Furthermore, a strengthening of the colourful Carathéodory Theorem, given in [1], states that if $\mathbf{0} \in \bigcap_{i=1}^{d+1} \text{conv}(\mathbf{S}_i)$, then each point of the colourful point configuration is in some colourful simplices containing $\mathbf{0}$ in their convex hulls. Hence, in an octahedral system Ω arising from such a colourful point configuration, each class V_i is covered.

2.2. Decompositions

The following proposition, proved in [6], states that the set of all octahedral systems is stable under the “symmetric difference” operation.

Proposition 1. *Let Ω and Ω' be two octahedral systems over the same vertex set. $\Omega \Delta \Omega'$ is an octahedral system.*

PROOF. Let $\Omega'' = \Omega \Delta \Omega'$. As Ω'' is a subset of $V_1 \times \cdots \times V_n$, we simply check that the parity condition is satisfied. Consider $X_1 \subseteq V_1, \dots, X_n \subseteq V_n$ with $|X_i| = 2$ for $i = 1, \dots, n$. We have

$$|\Omega'' \cap (X_1 \times \cdots \times X_n)| = |\Omega \cap (X_1 \times \cdots \times X_n)| + |\Omega' \cap (X_1 \times \cdots \times X_n)| - 2|\Omega \cap \Omega' \cap (X_1 \times \cdots \times X_n)|.$$

All the terms of the sum are even, which allows to conclude.

We now present a family of specific octahedral systems we call *umbrellas*. An umbrella U is a set of the form $\{x^{(1)}\} \times \cdots \times \{x^{(i-1)}\} \times V_i \times \{x^{(i+1)}\} \times \cdots \times \{x^{(n)}\}$, with $x^{(j)} \in V_j$ for $j \neq i$. The class V_i covered in U is called its *colour*. $T = (x^{(1)}, \dots, x^{(i-1)}, x^{(i+1)}, \dots, x^{(n)})$ is its *transversal*. An umbrella is clearly an octahedral system over $V_1 \times \cdots \times V_n$ and we have the following proposition.

Proposition 2. *Two umbrellas of the same colour have an edge in common if and only if they are equal.*

PROOF. An umbrella is entirely determined by its colour V_i and its transversal T . Therefore, if two umbrellas of the same colour have an edge in common, they necessarily have the same transversal, which implies that they are equal.

It was implicitly proved in Section 3 of [6] that any octahedral system can be described as a symmetric difference of umbrellas. In this paper, we describe an octahedral system as a symmetric difference of other octahedral systems to bound its cardinality. We now focus on octahedral systems where the size of each class is equal to the number of classes.

Consider a nonempty octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$ with $|V_i| = n$ for all $i \in \{1, \dots, n\}$. Denote by i_1 the smallest $i \in \{1, \dots, n\}$ such that V_i is covered in Ω and order the vertices $\{x_1, \dots, x_n\}$ of V_{i_1} by increasing degree: $\deg_\Omega(x_1) \leq \cdots \leq \deg_\Omega(x_n)$. We define \mathcal{U} to be the set of umbrellas of colour V_{i_1} containing an edge of Ω incident with x_1 and $W = \bigtriangleup_{U \in \mathcal{U}} U$. Let Ω_j be the set of all edges in $\Omega \Delta W$ incident with x_j . Formally,

$$\mathcal{U} = \{U : U \text{ umbrella of colour } V_{i_1} \text{ and } U \cap \delta_\Omega(x_1) \neq \emptyset\} \text{ and } \Omega_j = \delta_{\Omega \Delta W}(x_j).$$

Note that $|\mathcal{U}| = \deg_\Omega(x_1)$. In the remaining of the paper we refer to $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ as a *suitable decomposition*.

Lemma 2. Let $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ be a suitable decomposition and $W = \Delta_{U \in \mathcal{U}} U$. We have

- (i) $\Omega_j \cap \Omega_\ell = \emptyset$, for all $j \neq \ell$ (they have no edge in common),
- (ii) $\Omega = W \Delta \Omega_2 \Delta \dots \Delta \Omega_n$,
- (iii) Ω_j is an octahedral system, for all j ,
- (iv) $\deg_\Omega(x_j) \geq \max(|\mathcal{U}|, |\Omega_j| - |\Omega_j \cap W|)$ for all j .
- (v) If V_i is not covered in Ω , then V_i is neither covered in $\Omega \Delta W$ nor in any Ω_j .

The terminology suitable decomposition is due to point (ii) of Lemma 2.

PROOF (PROOF OF LEMMA 2). We first prove (i). The i_1 th component of any edge in Ω_j is x_j . Therefore, Ω_j and Ω_ℓ have no edge in common if $j \neq \ell$.

We then prove (ii). There are exactly $\deg_\Omega(x_1)$ umbrellas of colour V_{i_1} containing an edge of Ω incident with x_1 . As W is the symmetric difference of these umbrellas, x_1 is isolated in $\Omega \Delta W$. Thus, $\Omega_2, \dots, \Omega_n$ form a partition of the edges in $\Omega \Delta W$ and $\Omega \Delta W = \Omega_2 \Delta \dots \Delta \Omega_n$. Taking the symmetric difference of this equality with W we obtain $\Omega = W \Delta \Omega_2 \Delta \dots \Delta \Omega_n$.

We now prove (iii). By definition, the Ω_j 's are subsets of $V_1 \times \dots \times V_n$. It remains to prove that they satisfy the parity condition. Consider $X_i \subseteq V_i$ with $|X_i| = 2$ for $i = 1, \dots, n$. If X_{i_1} does not contain x_j , there are no edges in Ω_j induced by $X_1 \times \dots \times X_n$. If X_{i_1} contains x_j , the edges in Ω_j induced by $X_1 \times \dots \times X_n$ are the ones induced by $X_1 \times \dots \times X_{i_1-1} \times \{x_j\} \times X_{i_1+1} \times \dots \times X_n$. As x_1 is isolated in $\Omega \Delta W$, those edges are exactly the edges in $\Omega \Delta W$ induced by $X_1 \times \dots \times X_{i_1-1} \times \{x_1, x_j\} \times X_{i_1+1} \times \dots \times X_n$. According to Proposition 1, W is an octahedral system and $\Omega \Delta W$ as well, hence there is an even number of edges.

We prove (iv). We have $|\mathcal{U}| = \deg_\Omega(x_1) \leq \deg_\Omega(x_j)$ for all $j \in \{1, \dots, n\}$. Furthermore, by definition of the symmetric difference, we have $(\Omega_2 \Delta \dots \Delta \Omega_n) \setminus W \subseteq \Omega$. This inclusion becomes $(\Omega_2 \setminus W) \Delta \dots \Delta (\Omega_n \setminus W) \subseteq \Omega$. As two Ω_ℓ 's share no edges, $\Omega_j \setminus W \subseteq \Omega$ and thus $\Omega_j \setminus W \subseteq \delta_\Omega(x_j)$ for all $j \in \{2, \dots, n\}$. We obtain

$$|\Omega_j| - |\Omega_j \cap W| \leq \deg_\Omega(x_j).$$

Finally to prove (v) it suffices to prove that a class V_i not covered in Ω remains not covered in $\Omega \Delta W$. Indeed, if a class is covered in an Ω_j , it is also covered in $\Omega \Delta W$, as no two Ω_ℓ 's have an edge in common. Consider V_i not covered in Ω . There is a vertex $x \in V_i$ incident with no edges in Ω . In particular, there are no edges in Ω incident with x_1 and x . Therefore, the umbrellas in \mathcal{U} , which are defined by the edges incident with x_1 , contain no edges incident with x . Hence, x is isolated in $W = \Delta_{U \in \mathcal{U}} U$ and in Ω . Finally, x remains isolated in $\Omega \Delta W$.

Unlike the suitable decomposition of Ω , which is a decomposition over general octahedral systems, the decomposition given in the following lemma is over umbrellas.

Lemma 3. Consider an octahedral system $\Omega \subseteq V_1 \times \dots \times V_n$ with $|V_i| = n$ for all $i \in \{1, \dots, n\}$. There exists a set of umbrellas \mathcal{D} , such that $\Omega = \Delta_{U \in \mathcal{D}} U$ and such that the following implication holds:

$$V_i \text{ is the colour of some } U \in \mathcal{D} \implies V_i \text{ is covered in } \Omega.$$

PROOF. The proof works by induction on the number of covered classes in Ω . If no classes are covered, then, according to Lemma 1, Ω is empty.

Suppose now that k classes are covered, with $k \geq 1$, and consider a suitable decomposition $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ of Ω . Denote by W the symmetric difference $W = \Delta_{U \in \mathcal{U}} U$. According to Proposition 1, W is an octahedral system, and so is $\Omega \Delta W$. There are strictly fewer covered classes in $\Omega \Delta W$ than in Ω . Indeed, in $\Omega \Delta W$, the class V_{i_1} is no longer covered, since x_1 is isolated, and according to (v) of Lemma 2, a class not covered in Ω remains not covered in $\Omega \Delta W$. By induction, there exists a set \mathcal{D}' of umbrellas such that $\Omega \Delta W = \Delta_{U \in \mathcal{D}'} U$, and such that if there is an umbrella of colour V_i in \mathcal{D}' , then V_i is covered in $\Omega \Delta W$. As the umbrellas in \mathcal{D}' are not of colour V_{i_1} , we have $\mathcal{U} \cap \mathcal{D}' = \emptyset$. Therefore, $\Omega = (\Delta_{U \in \mathcal{U}} U) \Delta (\Delta_{U \in \mathcal{D}'} U)$ and the set $\mathcal{D} = \mathcal{U} \cup \mathcal{D}'$ satisfies the statement of the lemma.

3. Proof of the main result

The following theorem gives a general lower bound on the cardinality of an octahedral system. Our main theorem is a corollary of it.

Theorem 2. *Let $\Omega \subseteq V_1 \times \cdots \times V_n$ be an octahedral system with $|V_1| = \cdots = |V_n| = n \geq 2$. If $k \geq 1$ classes among the V_i 's are covered, then*

$$|\Omega| \geq k(n-2) + 2.$$

Before proving this theorem, we show how the main theorem can be deduced from it.

PROOF OF THEOREM 1. The inequality $\mu(d) \leq d^2 + 1$ is proved in [2]. Let $\mathbf{S}_1, \dots, \mathbf{S}_{d+1}$ be a colourful point configuration in \mathbb{R}^d . As explained in Section 2.1, the set $\Omega \subseteq V_1 \times \cdots \times V_{d+1}$, with $V_i = \mathbf{S}_i$ for $i = 1, \dots, d+1$ and whose edges correspond to the colourful simplices containing $\mathbf{0}$ in their convex hulls, is an octahedral system. According to [1, Theorem 2.3.], all the classes are covered in this octahedral system. Applying Theorem 2 with $k = n = d+1$ gives the lower bound: $\mu(d) \geq d^2 + 1$.

The remainder of the section is devoted to the proof of Theorem 2. The proof distinguishes two cases, corresponding to the following Propositions 3 and 4. We first prove these propositions.

Proposition 3. *Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$ with $|V_i| = n$ for all $i \in \{1, \dots, n\}$ and a class V_i covered in Ω . If Ω can be written as a symmetric difference of umbrellas, none of them being of colour V_i , then $|\Omega| \geq n^2$.*

PROOF. Let \mathcal{D} be a set of umbrellas such that there are no umbrellas of colour V_i in \mathcal{D} and $\Omega = \triangle_{U \in \mathcal{D}} U$. Denote by y_1, \dots, y_n the vertices of V_i , and by \mathcal{Q}_j the set of umbrellas in \mathcal{D} incident with y_j for each $j \in \{1, \dots, n\}$. As \mathcal{D} does not contain any umbrellas of colour V_i , the umbrellas in \mathcal{Q}_j all have transversals with i th component equal to y_j . Denote by Q_j the symmetric difference of the umbrellas in \mathcal{Q}_j . We have that Q_j is an octahedral system, according to Proposition 1, and that $\delta_\Omega(y_j) = Q_j$, $Q_j \neq \emptyset$, and $Q_j \cap Q_\ell = \emptyset$ for all $j \neq \ell$. According to Lemma 1, at least one class is covered in Q_j and hence $|Q_j| \geq n$. Therefore, we have

$$|\Omega| = \sum_{j=1}^n \deg_\Omega(y_j) = \sum_{j=1}^n |Q_j| \geq n^2$$

Proposition 4. *Consider an octahedral system $\Omega \subseteq V_1 \times \cdots \times V_n$ with $|V_i| = n$ for all $i \in \{1, \dots, n\}$ and a suitable decomposition $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ of Ω . Consider $\mathcal{O} \subseteq \{\Omega_2, \dots, \Omega_n\}$ such that for each $\Omega_j \in \mathcal{O}$ there is a class V_i covered in Ω_j and in no other $\Omega_\ell \in \mathcal{O}$. Denote by $\mathcal{P} \subseteq \mathcal{O}$ the set of umbrellas in \mathcal{O} . We have*

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} |\Omega_j| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

PROOF. Let $W = \triangle_{U \in \mathcal{U}} U$. The number of edges in Ω is equal to $\sum_{j=1}^n \deg_\Omega(x_j)$. We bound $\deg_\Omega(x_j)$ by $|\mathcal{U}|$ for $j = 1$ and if $\Omega_j \notin \mathcal{O}$ and by $|\Omega_j| - |\Omega_j \cap W|$ otherwise, see (iv) in Lemma 2. We obtain

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} (|\Omega_j| - |\Omega_j \cap W|).$$

We introduce a graph $G = (\mathcal{V}, \mathcal{E})$ defined as follows. We use the terminology *nodes* and *links* for G in order to avoid confusion with the vertices and edges of Ω . The nodes in \mathcal{V} are identified with the umbrellas in \mathcal{U} and the Ω_j 's in \mathcal{O} : $\mathcal{V} = \mathcal{U} \cup \mathcal{O}$. There is a link in \mathcal{E} between two nodes if the corresponding octahedral systems have an edge in common. G is bipartite: indeed, two umbrellas in \mathcal{U} are of the same colour V_{i_1} and, according to Proposition 2, they do not have an edge in common. According to Lemma 2, two Ω_j 's do not have an edge in common either.

For Ω_j in \mathcal{O} , we have $|\Omega_j \cap W| = \sum_{U \in \mathcal{U}} |\Omega_j \cap U| = \deg_G(\Omega_j)$, note that here the degree is counted in G . The fact that the umbrellas in \mathcal{U} are disjoint proves the first equality. The second equality is deduced from the facts that Ω_j has at most one edge in common with each umbrella in \mathcal{U} , the one incident with x_j , and that Ω_j has no neighbours in \mathcal{O} . We obtain the following bound

$$\begin{aligned} |\Omega| &\geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} (|\Omega_j| - \deg_G(\Omega_j)) \\ &= |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} |\Omega_j| - \deg_G(\mathcal{O} \setminus \mathcal{P}) - \deg_G(\mathcal{P}). \end{aligned}$$

Again, for the equality, we use the fact that G is bipartite. The number of links in \mathcal{E} incident with a node in $\mathcal{O} \setminus \mathcal{P}$ is at most $|\mathcal{U}|$. Hence, $\deg_G(\mathcal{O} \setminus \mathcal{P}) \leq |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|)$. It remains to bound $\deg_G(\mathcal{P})$. Note that if U is an umbrella in \mathcal{P} , it is the only umbrella of its colour in \mathcal{P} , otherwise it would contradict the property of \mathcal{O} . We now prove that there are no cycles induced by $\mathcal{P} \cup \mathcal{U}$ in G .

Suppose there is such a cycle \mathcal{C} and consider an umbrella U of \mathcal{P} in this cycle. Denote its colour by V_i and its neighbours in \mathcal{C} by L and R . As G is simple, L and R are distinct. L and R are both in \mathcal{U} , and hence are of colour V_{i_1} and do not have an edge in common. Therefore $U \cap L$ and $U \cap R$ do not have an edge in common either, which implies that the i th component of the transversals of L and R are distinct. Note that two umbrellas adjacent in \mathcal{C} , both of colour distinct from V_i , have necessarily transversals with the same i th component. Hence there must be another umbrella of colour V_i in the path in \mathcal{C} between L and R not containing U . This is a contradiction since U is the only umbrella in \mathcal{P} of colour V_i .

The number of links in \mathcal{E} incident with \mathcal{P} is then at most $|\mathcal{U}| + |\mathcal{P}| - 1$. This allows us to conclude.

PROOF (PROOF OF THEOREM 2). Let $\Omega \subseteq V_1 \times \dots \times V_n$ be an octahedral system with $|V_1| = \dots = |V_n| = n \geq 2$, and suppose that $k \geq 1$ classes V_{i_1}, \dots, V_{i_k} , with $i_1 < \dots < i_k$, are covered in Ω . The proof works by induction on k .

If $k = 1$, then Ω must contain at least n edges for one class to be covered.

Assume now that $k > 1$. If $|\mathcal{U}| \geq n - 1$, then, according to (iv) of Lemma 2, $|\Omega| = \sum_{j=1}^n \deg_\Omega(x_j) \geq n|\mathcal{U}| \geq k(n - 2) + 2$ and we are done. Assume now that $|\mathcal{U}| \leq n - 2$. We consider a suitable decomposition $(\mathcal{U}, \Omega_2, \dots, \Omega_n)$ of Ω and distinguish two cases.

Case 1: *One of the covered classes V_i , for $i \in \{i_2, \dots, i_k\}$, is not covered in any Ω_j .* Let V_i be a covered class in Ω , which is not covered in any Ω_j . For each $j \in \{2, \dots, n\}$, applying Lemma 3 on Ω_j gives a set \mathcal{D}_j of umbrellas, all of colour distinct from V_i , such that $\Omega_j = \Delta_{U \in \mathcal{D}_j} U$. We obtain $\Omega = (\Delta_{U \in \mathcal{U}} U) \Delta (\Delta_{j=2}^n \Delta_{U \in \mathcal{D}_j} U)$, according to (ii) of Lemma 2. Thus, we can apply Proposition 3 which ensures that

$$|\Omega| \geq n^2 \geq k(n - 2) + 2.$$

Case 2: *Each covered class V_i , for $i \in \{i_2, \dots, i_k\}$, is covered in at least one of the Ω_j .* Choose a set $\mathcal{O} \subseteq \{\Omega_2, \dots, \Omega_n\}$, minimal for inclusion, such that each covered class V_i , for $i \in \{i_2, \dots, i_k\}$, is covered in at least one of the $\Omega_j \in \mathcal{O}$. Such a set \mathcal{O} satisfies the statement of Proposition 4. Applying this proposition, we obtain

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \sum_{\Omega_j \in \mathcal{O}} |\Omega_j| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

We now bound $\sum_{\Omega_j \in \mathcal{O}} |\Omega_j|$. Let k_j be the number of classes covered in Ω_j . By minimality of \mathcal{O} , there is at least one class covered in each $\Omega_j \in \mathcal{O}$, and according to (v) of Lemma 2 we have $k_j < k$, hence $1 \leq k_j < k$. By induction, the cardinality of Ω_j is at least $k_j(n - 2) + 2$. This lower bound is not good enough for the $\Omega_j \notin \mathcal{P}$ such that $k_j = 1$. We denote by \mathcal{A} those Ω_j 's. We explain now how to improve the lower bound for $\Omega_j \in \mathcal{A}$. Only one class is covered in Ω_j and $\Omega_j \notin \mathcal{P}$. According to Lemma 3, Ω_j can be written as a symmetric difference of distinct umbrellas of the same colour. According to Proposition 2, these umbrellas are pairwise disjoint and $|\Omega_j|$ is equal to n times the number of umbrellas in this decomposition.

Since Ω_j is not an umbrella itself, otherwise Ω_j would have been in \mathcal{P} , there are at least two umbrellas in this decomposition. We obtain

$$\sum_{\Omega_j \in \mathcal{O}} |\Omega_j| \geq \left(\sum_{\Omega_j \in \mathcal{O} \setminus \mathcal{A}} k_j \right) (n-2) + 2|\mathcal{O} \setminus \mathcal{A}| + 2n|\mathcal{A}| = \left(\sum_{\Omega_j \in \mathcal{O}} k_j \right) (n-2) + 2|\mathcal{O}| + n|\mathcal{A}|$$

We have thus

$$|\Omega| \geq |\mathcal{U}|(n - |\mathcal{O}|) + \left(\sum_{\Omega_j \in \mathcal{O}} k_j \right) (n-2) + 2|\mathcal{O}| + n|\mathcal{A}| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1.$$

Finally, we have

$$2|\mathcal{O}| - |\mathcal{P}| - |\mathcal{A}| \leq \sum_{\Omega_j \in \mathcal{O}} k_j \quad (1)$$

$$k-1 \leq \sum_{\Omega_j \in \mathcal{O}} k_j \quad (2)$$

Equation (1) is obtained by distinguishing the Ω_j with $k_j = 1$ from those with $k_j \geq 2$. Equation (2) results from the fact that each class V_{i_2}, \dots, V_{i_k} is covered in at least one Ω_j in \mathcal{O} . Thus,

$$\begin{aligned} |\Omega| &\geq |\mathcal{U}|(n - |\mathcal{O}|) + \left(\sum_{\Omega_j \in \mathcal{O}} k_j \right) (n-2) + 2|\mathcal{O}| + |\mathcal{U}||\mathcal{A}| - |\mathcal{U}|(|\mathcal{O}| - |\mathcal{P}|) - |\mathcal{U}| - |\mathcal{P}| + 1 \\ &\geq (k-1)(n-2) + 2|\mathcal{O}| - |\mathcal{P}| + 1 + \left(\sum_{\Omega_j \in \mathcal{O}} k_j - k + |\mathcal{A}| + n - 2|\mathcal{O}| + |\mathcal{P}| \right) |\mathcal{U}| \end{aligned}$$

where we only used the inequalities $n \geq n-2 \geq |\mathcal{U}|$ and (2). According to (1), the expression

$$\left(\sum_{\Omega_j \in \mathcal{O}} k_j - k + |\mathcal{A}| + n - 2|\mathcal{O}| + |\mathcal{P}| \right)$$

is nonnegative. Moreover, we have already noted that $|\mathcal{U}| = \deg_{\Omega}(x_1)$, which is at least 1. Therefore,

$$|\Omega| \geq (k-1)(n-2) + 2|\mathcal{O}| - |\mathcal{P}| + 1 + \sum_{\Omega_j \in \mathcal{O}} k_j - k + |\mathcal{A}| + n - 2|\mathcal{O}| + |\mathcal{P}|.$$

Using (2) again, we obtain

$$|\Omega| \geq k(n-2) + 2.$$

Aknowlegement

The author thanks Antoine Deza for introducing her to the colourful simplicial depth conjecture and Frédéric Meunier for his thorough reading of the manuscript and his helpful comments.

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